UDC 539.376

ON NUMERICAL REALIZATION OF THE FUNCTION OF THE HEREDITARY OPERATOR

PMM Vol.42, No.6, 1978, pp. 1115-1122
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(Received March 28, 1978)

The problem of numerical realization of a function of the hereditary operator acting on some function of time is considered. Laplace transformations are used for the operators with kernels of the Rabotnov and Rzhanitsyn type to obtain formulas which reduce the problem in question to that of computing a quadrature. When the variable assumes large values, the formulas become asymptotic equations with an estimable error of approximation.

1. Effective solution of a wide class of problems of hereditary elasticity requires the application of the Volterra principle [1]. The replacement of the elastic constants of the material by the corresponding rheological operators, which is carried out in the solution of the problem for a perfectly elastic body, leads to the necessity of computing the convolutions of the form

$$\varphi(x; q^*) f(t), \quad q^* f(t) \equiv \int_0^t q(t-\tau) f(\tau)\, d\tau \tag{1.1}$$
$$q_j^* q_k^* f = q_j^* (q_k^* f), \quad q^{*^n} f = q^* (q^{*^{n-1}} f)$$

Here φ is a function of the spacial coordinates x_k and integral operators q_j^* in the sense adopted in [1], t is time, and $q(t-\tau)$ is a kernel of the operator q^*, depending on the difference of the arguments.

If φ is a rational function of the resolvent operators of the same class, then the expression (1.1) reduces to quadratures according to the well known rules [1]. In the general case the numerical realization of (1.1) is achieved by writing the function φ in the form of a series in powers of the operators q^*, and then applying the formulas (1.1). A detailed interpretation of the functions of the operators and consequent computations are, as a rule, very time-consuming except in the case of small t, the latter ensuring the rapid convergence of the series. It is therefore expedient to construct effective computational algorithms for the expressions (1.1).

When Laplace transforms are applied to the initial relationships of the theory of hereditary elasticity we find, that in order to compute (1.1), it is necessary to invert the expression $\varphi(x; Q(p))F(p)$ where p is the transformation parameter, $Q(p) = L\{q(t)\}$ and $F(p) = L\{f(t)\}$ are the Laplace transforms of the kernel $q(t)$ of the operator q^* and of the function $f(t)$ respectively. Applying this to the operators q^* with the hereditary kernels appearing as the fractional power exponents $\vartheta_\alpha(\beta, t)$ due to Rabotnov [1] and the function $P_\alpha(\lambda, t)$ due to Rzhanitsyn [2]

$$\vartheta_\alpha(\beta, t) = \sum_{k=0}^{\infty} \frac{\beta^k t^{kr+\alpha}}{\Gamma[r(k+1)]}, \quad \beta < 0, \quad 0 < r = 1 + \alpha < 1 \tag{1.2}$$

$$P_\alpha(\lambda, t) = \frac{1}{\Gamma(r)} t^\alpha e^{\lambda t}, \quad \lambda \leqslant 0, \quad 0 < r = 1 + \alpha < 1 \tag{1.3}$$

the Laplace transforms of which are analytic functions with a characteristic singularity in the form of a branch point on the complex p-plane, we can compute the convolutions of the type (1.1) over a wide range of values of t.

2. Let φ be a function of a single opeator q^* and

$$Q(p) = Q_1(z), \; z = (p - \lambda)^r, \; \lambda \leqslant 0, \; 0 < r < 1 \tag{2.1}$$

According to the Mellin formula,

$$\varphi(q^*) f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \varphi(Q(p)) F(p) e^{pt} dp \tag{2.2}$$

$$\varphi(q^*) \equiv \varphi(x; \; q^*), \; \varphi(Q(p)) \equiv \varphi(x; \; Q(p))$$

Here the integration is carried out along the straight line $\operatorname{Re} p = \sigma$ situated in the p-plane to the right of all singularities of the integrand function. We shall assume that in the finite part of the plane, with $|\arg p| \leqslant \pi$, this function can only have a single branch point $p = \lambda$ and, perhaps, a finite number of poles, none of them lying on the negative real semi-axis to the left of the branch point. To separate a single-valued branch of the function, we produce a cut along this semi-axis to the left of the point $p = \lambda$. Taking into account the residues of the poles, we replace the integration path in (2.2) by a contour consisting of the upper Γ_+ and lower Γ_- edge of the cut, the arcs Γ_1 and Γ_2 of the circle of infinite radius and of the circle Γ_ε of vanishingly small radius, with centers at the point $p = \lambda$. Assuming that the conditions of the Jordan lemma hold and the integrals along the arcs Γ_1 and Γ_2 vanish, we obtain

$$\varphi(q^*) f(t) = \frac{1}{2\pi i} \int_H \varphi(Q(p)) F(p) e^{pt} dp + \sum_s \operatorname*{Res}_{p_s} [\varphi(Q(p)) F(p) e^{pt}] \tag{2.3}$$

Here the contour $H = \Gamma_- + \Gamma_\varepsilon + \Gamma_+$ is traversed in the anticlockwise direction, and the residues are calculated over all poles p_s, $|\arg(p_s - \lambda)| \neq \pi$.

Substitution of $z = (p - \lambda)^r$ from (2.1) maps the contour H onto the contour H_1 in the z-plane, the latter contour consisting of the arcs $\arg z = \pm \pi r$ and the arc of the circle $|z| = \varepsilon^r = \varepsilon_1$. Let

$$\varphi(Q(p)) = \varphi_1(z) = z^{-\nu} \Phi(z)$$

Here $\nu \geqslant 0$, and $\Phi(z)$ is a single-valued function analytic on the contour H_1 and in the region $|z| \leqslant \varepsilon_1$. In this case we have the following approximation [3] on H_1:

$$\Phi(z) = \sum_{k=0}^{n} \Phi^{(k)}(0) \frac{z^k}{k!} + \rho_n(z), \quad \Phi^{(k)}(z) = \frac{d^k \Phi}{dz^k}, \quad n \geqslant 0 \tag{2.4}$$

$$|\rho_n(z)| \leqslant M'_{n+1} \frac{|z|^{n+1}}{(n+1)!}, \quad M'_{n+1} = \max_l \left| \frac{d^{n+1}}{dz^{n+1}} \Phi(z) \right| \tag{2.5}$$

where l is a line connecting the point z of the contour H_1 with the point $z = 0$.

We assume that the following representation is possible for the function $F(p)$ analytic on H, but not necessarily regular at the point $p = \lambda$, for all $p \in H$:

$$F(p) = \sum_{j=0}^{m} b_j (p-\lambda)^{j+\mu} + \gamma_m, \quad m \geqslant 0, \quad \mu \gtrless 0 \tag{2.6}$$

$$|\gamma_m| \leqslant A\, |p-\lambda|^{m+\mu+1}, \qquad A = \text{const} \geqslant 0 \tag{2.7}$$

The conditions (2.6) and (2.7) are satisfied, in particular, by the transforms of the functions t^δ $(\delta > -1)$, $\sin \omega t$, $\cos \omega t$, e^{ht} $(h \geqslant \lambda)$.

By virtue of the equation [4]

$$\frac{1}{2\pi i} \int_H (p-\lambda)^\omega e^{pt}\, dp = T_{\text{I}}(t;\omega)\, e^{\lambda t}, \quad T_{\text{I}}(t;\omega) \equiv \frac{t^{-\omega-1}}{\Gamma(-\omega)}, \quad \omega \gtrless 0 \tag{2.8}$$

the expansions (2.4) and (2.6) transform (2.3) to the form

$$\varphi(q^*) f(t) = e^{\lambda t} \sum_{k=0}^{n} \sum_{j=0}^{m} b_j \frac{\Phi^{(k)}(0)}{k!} T_{\text{I}}(t; rk + j + \mu - r\nu) + \tag{2.9}$$

$$\sum_s \operatorname*{Res}_{p_s} [\varphi(Q(p)) F(p) e^{pt}] + r_{mn}(t)$$

$$r_{mn}(t) = \tag{2.10}$$

$$\frac{1}{2\pi i} \left(\int_{\Gamma_-} + \int_{\Gamma_\varepsilon} + \int_{\Gamma_+} \right) \left\{ e^{pt} \left[\gamma_m \varphi(Q(p)) + \rho_n \sum_{j=0}^{m} b_j (p-\lambda)^{j+\mu-r\nu} \right] dp \right\}$$

If the numbers n and m in (2.4) and (2.6) satisfy the inequalities

$$r(n+1) + \mu > r\nu - 1, \quad m + \mu + 1 > r\nu - 1 \tag{2.11}$$

then the integral along Γ_ε in (2.10) vanishes as $\varepsilon \to 0$ by virtue of (2.5) and (2.7) and the boundedness of the function $\Phi(z)$ near the point $z = 0$. The remaining integrals along the edges Γ_+ and Γ_- of the cut on which $p - \lambda = xe^{\pm i\pi}$, $x = |p-\lambda|$, yield the formula

$$r_{mn}(t) = -\frac{1}{\pi} e^{\lambda t} \int_0^\infty e^{-xt} V(x)\, dx \tag{2.12}$$

Here $V(x)$ denotes the imaginary part of the expression within the square brackets in (2.10) at the upper edge of the cut, i.e.

$$V(x) = \mathrm{Im}\left[\gamma_m \varphi(Q(p)) + \rho_n \sum_{j=0}^{m} b_j (p-\lambda)^{j+\mu-r\nu}\right]_{p=\lambda+xe^{i\pi}} \tag{2.13}$$

We note that if $\rho_n = 0$ or $\gamma_m = 0$, then the first or second condition of (2.11), becomes respectively superfluous.

Taking into account the relations (2.5) and (2.7) we obtain, from (2.12) and (2.13), the following estimate (where the maximum is taken over $\arg z = \pm \pi r$) :

$$|r_{mn}(t)| \leqslant \frac{1}{\pi} e^{\lambda t} \Big\{ A M_0 T_2(t; m+\mu-r\nu+2) + \tag{2.14}$$

$$\frac{M_{n+1}}{(n+1)!} \sum_{j=0}^{m} |b_j| T_2(t; rn+r+j+\mu-r\nu+1) \Big\}$$

$$T_2(t;\omega) \equiv \Gamma(\omega) t^{-\omega}, \quad M_0 = \max|\Phi(z)|, \quad M_{n+1} = \max\left|\frac{d^{n+1}\Phi(z)}{dz^{n+1}}\right|$$

When the values of t are sufficiently large, then (2.14) implies that $r_{mn}(t)$ can be neglected and the expression (2.9) used as an asymptotic expansion. Moreover, the formulas (2.9) and (2.12) make possible the calculation of the convolution $\varphi(q^*)f(t)$ also for other values of t, provided that the quantities ρ_n and γ_m in (2.13) are expressed in terms of the relations (2.4) and (2.6) and an approximate value of the quadrature (2.12) is found. The numbers n and m should be chosen as small as possible, with (2.11) taken into account. The integral in (2.12) is a Laplace transform, with parameter t, of the function of real variable $V(x)$. This proves the following theorem.

Theorem. Let

(1) a function $\varphi(u)$ analytic in the circle $|u| \leqslant u_0$ ($u_0 =$ const > 0) *define a function of the hereditary operator* $\varphi(q^*)$ (1.1) and let the Laplace transform $Q(p)$ of the operator q^* on the plane of the transformation parameter have a branch point $p = \lambda$ ($\lambda \leqslant 0$) and $\varphi(Q(p)) = (p-\lambda)^{-r\nu}\Phi[(p-\lambda)^r]$, $0 < r < 1$, $\nu \geqslant 0$ where $\Phi(z)$ is a single-valued function on the rays $\arg z = \pm \pi r$, at the point $z = 0$ and in its neighborhood;

(2) the operator function $\varphi(q^*)$ act on the function $f(t)$ the Laplace transform $F(p)$ of which admits, on the negative semiaxis, at $p < \lambda$ and near the point $p = \lambda$, the representation (2.6), (2.7):

(3) the product $\varphi(Q(p))F(p)$ tend to zero uniformly in $\arg p$ as $p \to \infty$ and $\mathrm{Re}\, p < \sigma =$ const ;

(4) amongst the singularities of the expression $\varphi(Q(p))F(p)$ there exists, on a finite part of the p-plane with $|\arg p| \leqslant \pi$, only a single branch point $p = \lambda$ and possibly a finite number of poles none of which lie on the negative part of the real semi-axis to the left of the point $p = \lambda$.

Then the convolution $\varphi(q^*)f(t)$ will be given, for any finite $t > 0$, by the formulas (2.9), (2.12) and (2.13), and at sufficiently large t the relations (2.9) will assume the meaning of an asymptotic equation with (2.14) providing the estimate of its remainder term.

The formula (2.9) enables us, in particular, to establish the behavior of the convolution $\varphi(q^*)f(t)$ as $t \to \infty$. A finite limit value exists if $\mathrm{Re}\, p_s < 0$ and either $\lambda < 0$ or $\lambda = 0$ for all poles p_s at which the residues are computed without however positive powers of t appearing in the expansion (2.9). When $\lambda = 0$ and $f(t) = \text{const}$, the index $\mu = -1$ and, provided that the conditions $\operatorname{Re} p_s < 0$, $\Phi(0) \neq 0$ hold, the convolution is bounded if $\nu = 0$.

From the known theorem of operational calculus we have

$$\lim_{t\to\infty} q^* \cdot 1 = \lim_{p\to 0} p\left[Q(p)\frac{1}{p}\right] = Q(0) \tag{2.15}$$

$$\lim_{t\to\infty} \varphi(q^*) f(t) = \lim_{p\to 0} p\varphi(Q(p)) F(p) = \varphi(Q(0)) \lim_{p\to 0} pF(p) = \varphi(\lim_{t\to\infty} q^* \cdot 1) f(\infty)$$

provided that the limits in question exist. The assertion (2.15) represents a limiting theorem for the hereditary operators, established in [1].

3. Let us now consider the operator q^* with the kernel (1.2) $\varkappa \partial_\alpha(\beta, t), \varkappa > 0$. Since $L\{\partial_\alpha(\beta, t)\} = (p^r - \beta)^{-1}$, we must put $\lambda = 0$ and $z = p$ and assume, in the case when $\varphi(Q(p))$ has no singularity at the point $p = 0$, that $\nu = 0$. We also have

$$\Phi(z) = \varphi(\varkappa/(z-\beta)),\ \Phi^{(k)}(0) = (-1)^k \partial^k \varphi(-\varkappa/\beta)/\partial\beta^k \tag{3.1}$$

$$\varphi(\varkappa \partial_\alpha^*(\beta)) f(t) = \sum_{k=0}^{n}\sum_{j=0}^{m} b_j \frac{(-1)^k}{k!} \frac{\partial^k \varphi(-\varkappa/\beta)}{\partial\beta^k} T_1(t; rk + j + \mu) + \sum_s \operatorname*{Res}_{p_s}\left[\varphi\left(\frac{\varkappa}{p^r - \beta}\right) F(p) e^{pt}\right] + r_{mn}(t)$$

Let the function $\varphi(u)$ be analytic in the circle $|u| \leqslant u_0$ and

$$\max_{\arg z = \pm\pi r}\left|\frac{\varkappa}{z-\beta}\right| = \frac{\varkappa}{g} < u_0$$

Then, when $r \leqslant 1/2$ $g = |\beta|$ and $r > 1/2$ $g = |\beta| \sin \pi r$. In this case the quantities M_0 and M_{n+1} from (2.14) can be written in the form of a series. In particular, for M_{n+1} we have

$$M_{n+1} = \max_{\arg z = \pm\pi r}\left|\sum_{j=0}^{\infty}(-1)^{n+1} S_{jn}(z-\beta)^{-j-n-1}\right|$$

$$S_{jn} \equiv (j+n)!\, \varkappa^j \varphi^{(j)}(0)/[(j-1)!\, j!]$$

If all $\varphi^{(j)}(0)$, $j = 0, 1, 2, \ldots$, have the same sign, then

$$M_{n+1} \leqslant \left|\sum_{j=0}^{\infty} S_{jn} g^{-j-n-1}\right| = \left|\frac{\partial^{n+1}\varphi(\varkappa/g)}{\partial g^{n+1}}\right|, \quad M_0 \leqslant \left|\varphi\left(\frac{\varkappa}{g}\right)\right| \tag{3.2}$$

When the sign alternates, i.e., $\operatorname{sign} \varphi^{(j)}(0) = -\operatorname{sign} \varphi^{(j+1)}(0)$, $j = 0, 1, 2, \ldots$, we have

$$M_{n+1} \leqslant \Big| \sum_{j=0}^{\infty} (-1)^j S_{jn} g^{-j-n-1} \Big| = \Big| \frac{\partial^{n+1} \varphi(-\varkappa/g)}{\partial g^{n+1}} \Big|, \quad M_0 \leqslant \Big| \varphi\Big(-\frac{\varkappa}{g}\Big) \Big| \quad (3.3)$$

The above inequalities simplify the estimation of $r_{mn}(t)$ when formula (2. 14) is used.

When $f(t) = \text{const}$ and $\lambda = 0$ in (2. 6), we have $b_0 \neq 0$, $b_j = 0$ $(j \geqslant 1)$, $\mu = -1$, $\gamma_m = 0$, and the formula (3. 1) coincides, in the absence of the residues, with the asymptotic expansion obtained in [5]. If $f(t) = t^{\delta}$ $(\delta > -1)$, $\varphi(q^*) = q^{*\omega}$, $\omega = 1, 2, \ldots$, $q(t) = \varkappa \vartheta_\alpha(\beta, t)$, then $b_0 = \Gamma(1 + \delta)$, $b_j = 0$ $(j \geqslant 1)$, $\mu = -\delta - 1$, $\gamma_m = 0$. In this case the formula (3. 1) and the estimates (2. 14) and (3. 2) yield the relations obtained in [6].

Example 1. Let $\varphi(\varkappa\vartheta_\alpha^*(\beta)) = (1 + \varkappa\vartheta_\alpha^*(\beta))^{1/2}$, $\varkappa > 0$, $\beta < 0$, $f(t) = 1$. Then $F(p) = p^{-1}$ and in (2. 6) we have $\lambda = 0$, $b_0 = 1$, $m = 0$, $\mu = -1$, $\gamma_m = 0$. When $n = 1$, we have, in accordance with (3. 1),

$$(1 + \varkappa\vartheta_\alpha^*(\beta))^{1/2} \cdot 1 = \Big(1 - \frac{\varkappa}{\beta}\Big)^{1/2} - \frac{\varkappa}{2\beta^2 (1 - \varkappa\beta^{-1})^{1/2}} \frac{t^{-r}}{\Gamma(1 - r)} + r_1(t)$$

From (2. 14) we obtain

$$| r_1(t) | \leqslant M_2 \Gamma(2r) t^{-2r} / (2\pi), \quad M_2 \leqslant \varkappa g^{-5/2} g_1^{-1/2} (\varkappa / (4g_1) + 1)$$

Here, when $r \leqslant {}^1/_2\, g = |\beta|, g_1 = |\beta_1|$, and when $r > {}^1/_2\ g = |\beta| \sin \pi r, g_1 = |\beta_1| \sin \pi r$, $\beta_1 = \beta - \varkappa$. Thus, if $\alpha = -0.7$ $(r = 0.3)$, $\beta = -1$, $\varkappa = 0.5$, then $(1 + \varkappa\vartheta_\alpha^*(\beta))^{1/2} \cdot 1 = 1.225 - 0.157 t^{-0.3} + r_1(t)$ and $| r_1(t) | \leqslant 0.105 t^{-0.6}$. The above results can be extended to the case of two and more operator arguments of the function φ. In particular, when $\varphi = \varphi(\varkappa_1\vartheta_{\alpha_1}^*(\beta_1), \varkappa_2\vartheta_{\alpha_2}^*(\beta_2))$ and under the condition that the operational analog of this function is bounded in the neighborhood of the point $p = 0$ and $1 + \alpha_1 = c_1 r$, $1 + \alpha_2 = c_2 r$ where c_1 and c_2 are positive integers and $0 < r < 1$, we have the following relations for the function $\Phi(z)$:

$$\Phi(z) = \varphi(\varkappa_1 / (z^{c_1} - \beta_1), \quad \varkappa_2 / (z^{c_2} - \beta_2)), \quad z = p^r$$

Using similar arguments, we obtain

$$\varphi(\varkappa_1\vartheta_{\alpha_1}^*(\beta_1), \varkappa_2\vartheta_{\alpha_2}^*(\beta_2)) f(t) = \sum_{k=0}^{n} \sum_{j=0}^{m} b_j \frac{\Phi^{(k)}(0)}{k!} T_1(t; rk + j + \mu) + \quad (3.4)$$

$$\sum_s \operatorname*{Res}_{p_s} \Big[\varphi \Big(\frac{\varkappa_1}{p^{c_1 r} - \beta_1}, \frac{\varkappa_2}{p^{c_2 r} - \beta_2} \Big) F(p) e^{pt} \Big] + r_{mn}(t)$$

When $c_1 = c_2 = 1$, the following relation holds:

$$\Phi^{(k)}(0) = (-1)^k (\partial / \partial\beta_1 + \partial / \partial\beta_2)^k \varphi(-\varkappa_1 / \beta_1, -\varkappa_2 / \beta_2)$$

The estimate of $r_{mn}(t)$ retains the form of (2. 14) when $\nu = 0, \lambda = 0$.

Example 2. The operator expression $(1 + \varkappa_1\vartheta_\alpha^*(\beta_1) + \varkappa_2\vartheta_\alpha^*(\beta_2))^{1/2} \cdot 1$, $\varkappa_1, \varkappa_2 > 0$, $\beta_1, \beta_2 < 0$, has, according to (3. 4), the following representation:

$$(1 + \varkappa_1\vartheta_\alpha^*(\beta_1) + \varkappa_2\vartheta_\alpha^*(\beta_2))^{1/2} \cdot 1 = \Big(1 - \frac{\varkappa_1}{\beta_1} - \frac{\varkappa_2}{\beta_2}\Big)^{1/2} -$$

$$\frac{\varkappa_1\beta_1^{-2}+\varkappa_2\beta_2^{-2}}{2\,(1-\varkappa_1\beta_1^{-1}-\varkappa_2\beta_2^{-1})^{1/2}}\,\frac{t^{-r}}{\Gamma\,(1-r)}+r_1\,(t)$$

From (2.14) we obtain

$$|\,r_1\,(t)\,|\leqslant\frac{M_2\Gamma\,(2r)}{2\pi}\,t^{-2r},\qquad M_2\leqslant\left(1+\frac{|\,\beta_1-\beta_2\,|}{g}\right)^{1/2}\times$$
$$\left(\frac{\varkappa^2}{g^4}+\frac{\varkappa^2}{2g^5}\,|\,\beta_1-\beta_2\,|+\frac{2\varkappa}{g^3}\right)$$

$r=1+\alpha,\quad \varkappa=\max\{\varkappa_1,\varkappa_2\},\qquad g=\min\{g_j\},\qquad j=1,2,3,4;\quad r\leqslant{}^1/_2\quad g_j=|\,\beta_j|$
$r>{}^1/_2\quad g_j=|\,\beta_j\,|\sin\pi r;\qquad \beta_{3,4}=-{}^1/_2\,[e_1+e_2\pm((e_1-e_2)^2+4\varkappa_1\varkappa_2)^{1/2}]$
$(e_1=\varkappa_1-\beta_1,\ e_2=\varkappa_2-\beta_2)$

Asymptotic expansion of the function of the operator q^* with kernel (1.3) is obtained in the analogous manner.

Example 3. Let $\varphi\,(q^*)f\,(t)=q^{*\omega}\cdot 1,\ \omega=1,2,\ldots,q\,(t)=\varkappa P_\alpha\,(\lambda,t),\ \varkappa>0,\ \lambda<0.$ Then

$$\varphi\,(Q\,(p))=\varkappa^{\omega}z^{-\omega},\quad z=(p-\lambda)^r,\quad \Phi\,(z)=\Phi\,(0)=\varkappa^{\omega},\quad \nu=\omega$$

$$F\,(p)=\frac{1}{p}=\sum_{j=0}^{m}(-1)^j\lambda^{-j-1}\,(p-\lambda)^j+(-1)^{m+1}p^{-1}\lambda^{-m-1}\,(p-\lambda)^{m+1}$$

Thus we have, in accordance with the expressions (2.4), (2.6), (2.7) and (2.14), $n=0,\ \rho_n=0,\ \mu=0,\ A=|\,\lambda\,|^{-m-2},\ M_0=\varkappa^{\omega},\ M_{n+1}=0$, and from (2.9), (2.14) follows

$$\varkappa^{\omega}P_{\alpha}^{*\omega}\,(\lambda)\cdot 1=\varkappa^{\omega}e^{\lambda t}\sum_{j=0}^{m}(-1)^j\lambda^{-j-1}T_1\,(t;\,j-r\omega)+\tag{3.5}$$
$$\frac{\varkappa^{\omega}}{(-\lambda)^{r\omega}}+r_{mn}\,(t)$$
$$|\,r_{mn}\,(t)\,|\leqslant\varkappa^{\omega}\,|\,\lambda\,|^{-m-2}e^{\lambda t}T_2\,(t;\,m-r\omega+2)\,/\,\pi,\quad m-r\omega+2>0$$

If $r\omega$ is an integer, then the branch point of the function

$$\varkappa^{\omega}\,/\,[p\,(p-\lambda)^{r\omega}\,]=L\{\varkappa^{\omega}P_{\alpha}^{*\omega}\,(\lambda)\cdot 1\}$$

at $p=\lambda$ is replaced by a pole, and this simplifies the expression (3.5) since in this case $r_{mn}\,(t)=0$ and we have $T_1\,(t;\,j-r\omega)=0$ when $j\geqslant r\omega$.

4. Using (2.12), we construct an analytic expression for approximating the quantity $r_{mn}\,(t)$. Let us approximate the function $V\,(x)$ using combinations of the exponential expressions and power functions. Since the multiplier e^{-xt} decays rapidly, it is sufficient to attain a good approximation to $V\,(x)$ at the initial part of the interval of integration, ensuring at large x only that the manner of behavior is similar.

Let

$$V(x) \approx \sum_j a_j x^{\delta_j} e^{\lambda_j x}, \quad \delta_j > -1, \quad \lambda_j \leqslant 0$$

Then

$$\int_0^\infty e^{-xt} V(x)\,dx = \sum_j a_j \frac{\Gamma(\delta_j+1)}{(t-\lambda)^{\delta_j+1}} + \int_0^\infty e^{-xt}\left[V(x) - \sum_j a_j x^{\delta_j} e^{\lambda_j x}\right] dx$$

The last integral can be used to assess the error of approximation.

To illustrate this, let us consider the convolution $\partial_\alpha^*(\beta)\cdot 1$ the Laplace transform of which $(p^r-\beta)^{-1}p^{-1}$ for $0<r<1$, $\beta<0$, $|\arg p| \leqslant \pi$ has a unique singularity, namely a branch point at $p=0$. Since $\lambda=0$ and $F(p)=p^{-1}$, therefore we have in (2.6) $b_0=1$, $m=0$, $\mu=-1$ and $\gamma_m=0$. Here $\Phi(z)=(z-\beta)^{-1}$ and $\nu=0$, and to fulfil the conditions (2.11) it is sufficient to put $n=0$, which yields $\rho_n = \beta^{-1}p^r(p^r-\beta)^{-1}$. In accordance with (2.9), (2.12) and (2.13), we obtain [7]

$$\partial_\alpha^*(\beta)\cdot 1 = -\frac{1}{\beta} + \frac{\sin \pi r}{\pi\beta}\int_0^\infty e^{-x|\beta|^{1/r}t}\frac{x^{r-1}dx}{x^{2r}+2x^r\cos\pi r+1} \tag{4.1}$$

Let $r=0.3$ $(\alpha=-0.7)$. We interpolate the decaying function $x^{r-1}(x^{2r}+2x^r\cos\pi r+1)^{-1}$ on the interval [0, 4] using the expression

$$x^{r-1}e^{-0.3x}(1-1.18x^r+0.587x^{2r}+0.02x^{3r})$$

From (4.1) follows

$$\partial^*_{-0.7}(\beta)\cdot 1 \approx -\beta^{-1} + (\pi\beta)^{-1}\sin\pi r\sum_{j=1}^4 A_j\Gamma(jr)\,y^j \qquad A_1=1, \tag{4.2}$$

$$A_2=-1.18,\quad A_3=0.587,\quad A_4=0.02,\quad y=(\tau+0.3)^{-r},\quad \tau=|\beta|^{1/r}t$$

Comparison of the values computed with help of the formula (4.2) with those given in tables [8] shows that at $\tau=0.1$ the error is about 5%, at $\tau=0.2$ it does not exceed 1.5% and at $\tau \geqslant 0.3$ the values practically coincide. It must however be remembered that the asymptotic expansion of $\partial_\alpha^*(\beta)\cdot 1$ [8] hold only when $\tau \gg 1$.

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Translated by L. K.